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Padé-type approximants of entire functions and acceleration of convergence: Two examples

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Abstract

In this paper we deal with Padé-type approximants of entire functions. On two examples, we take the question: do these rational approximants converge more quickly than the partial sums to the entire functions? In particular, it is proved that the Padé-type approximants $(k-1,k)$ of $\exp(-z)$, whose generating polynomials are $(z+1/k)^k$, accelerate the convergence of the partial sums $\sum_{j=0}^{k-1}(1/j!)(-z)^j$.

Keywords: Entire functions; Padé-type approximants; Generating polynomial; Acceleration of convergence

0. Introduction

Let us consider the Padé-type approximants [1] $(k-1/k)$ of an entire function $f(z)=\sum_{i\geqslant 0}c_iz^i$. General theoretical results concerning the choice of the generating polynomials were obtained in [3]. Particularly this author proved the following theorem.

Theorem A. *If f is an entire function and if the generating polynomials are taken as $v_k=\prod_{i=1}^{k}(z-a_i)a_i\in\mathbb{C}$, then if $\lim_{i\to\infty}a_i=0$, $\lim_{k\to\infty}(k-1/k)_f(z)=f(z)$ uniformly on any compact subset of $\mathbb{C}$.*

Moreover, $\forall z\in\mathbb{C}$

$$\lim_{k\to\infty}|f(z)-(k-1/k)_f(z)|^{1/(k-1)}=0.$$

The results of this theorem are still valid if $v_k(z)=(z-a_k)^k$, $a_k\in\mathbb{C}$ with $\lim_{i\to\infty}a_i=0$. A review of all those results can be found in [2, pp. 77–79]. As an important example we have the exponential function e^{-z} with $a_k=-1/k$. This case was studied by several authors. We can mention [7, 8, 10]. Such approximations have also been studied in [9] but with different numerator. All these

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papers are concerned with convergence or uniform convergence on the positive real axis. On the other hand, Van Iseghem [11] showed that the sequence of the above Padé-type approximants converges uniformly and geometrically on every compact subset of the plane.

Our aim is to find results about convergence acceleration. More precisely, the purpose of the present work is to approach the following question: *Do the Padé-type approximants, whose generating polynomials are chosen as above, accelerate the convergence of the partial sums of the series* $\sum_{i \geqslant 0} c_i z^i$?

This paper is organized as follows. The computation of the approximants $(k-1/k)_f$ needs a knowledge of $c_0, c_1, \ldots, c_{k-1}$. Thus we must compare the two errors $|f(z)-(k-1/k)_f(z)|$ and $|f(z)-\sum_{i=0}^{k-1} c_i z^i|$. In Section 1 we deal with the Padé-type approximants $(k-1/k)$ of the function e^{-z} with generating polynomials $(z+1/k)^k$. We prove that the sequence of Padé-type approximants converges faster to $\exp(-z)$ than the partial sums of the Taylor series. Section 2 contains an explicit expression for $f(z)-(k-1/k)$. This is used to give a new proof of the result $\lim_{k\to\infty} |f(z)-(k-1/k)_f(z)|^{1/(k-1)} = 0$ in the case of $f(z) = \sum_{n=0}^{+\infty} q^{n(n-1)/2} z^n$ $(0<q<1)$ and $v_k(z) = (z-q^k)^k$. Finally we investigate the behaviour of the ratio

$$\left\{\frac{f(z)-(k-1/k)}{f(z)-\sum_{i=0}^{k-1} q^{i(i-1)/2} z^i}\right\}.$$

1. Padé-type approximants of e^{-z} whose denominators are $(1+z/k)^k$

Let us consider the Padé-type approximant $(k-1/k)$ of $z \mapsto e^{-z}$ with generating polynomial $(z+1/k)^k$. As is told in the Introduction, these approximants were studied by Van Iseghem. Particularly Van Iseghem recovered the results of Theorem A and gave an expression of $(k-1/k)$ with the help of Laguerre's polynomials (see also [7]).

1.1.

Let us consider now the error

$$E_k(z) = e^{-z} - (k-1/k)(z).$$

In [7] expressions of this error are given (see Theorem 4.2). However our purpose is to obtain an acceleration result. Thus, we have to find a very good estimate of E_k. Let us now give a new majorization of $|E_k(z)|$ that improves those given in [11]. We need the following lemma.

Lemma 1.1 (Brezinski [1, p. 21]). *Let us assume that f is holomorphic in some simply connected domain D' of the complex plane which contains the origin. Let C' be a simple closed path in this domain. If we put $C = \psi(C')$ where $\psi : x \mapsto 1/x$, we obtain for t belonging to the interior of C'*

$$f(t) - (k-1/k)_f(t) = \frac{1}{2i\pi}\frac{t^k}{\tilde{v}(t)}\int_C \frac{x^{-1} f(x^{-1}) v(x)}{1-xt}\,dx$$

(v is the generating polynomial and $\tilde{v}(t) = t^k v(t^{-1})$).

Theorem 1.2. *Let t be a given complex number. The following inequality is true for $k \in \mathbb{N}$ large enough*:

$$|E_k(t)| \leqslant 4\mathrm{e}^{|t|} \times |t|^k \times \left[\frac{\sqrt{\mathrm{e}}}{k}\right]^k.$$

Proof. Letting $r > 0$ such that $|t| < r$, by Lemma 1.1 we have

$$E_k(t) = \frac{1}{2\mathrm{i}\pi}\frac{t^k}{\tilde{v}_k(t)}\int_C \frac{x^{-1}f(x^{-1})v(x)}{1-xt}\,\mathrm{d}x$$

with $C = \{x \mid |x| = 1/r\}$ $(x = \mathrm{e}^{\mathrm{i}\theta}/r;\ \theta \in [0, 2\pi])$ and $v_k(x) = (x + 1/k)^k$,

$$\int_C \frac{x^{-1}f(x^{-1})v(x)}{1-xt}\,\mathrm{d}x = \int_0^{2\pi} \frac{r\mathrm{e}^{-\mathrm{i}\theta} \times \mathrm{e}^{-r\mathrm{e}^{-\mathrm{i}\theta}} \times (\mathrm{e}^{\mathrm{i}\theta}/r + 1/k)^k \times (\mathrm{i}/r)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{1 - (\mathrm{e}^{\mathrm{i}\theta} \times t)/r}$$

$$= \mathrm{i}\int_0^{2\pi} \frac{\mathrm{e}^{-r\mathrm{e}^{-\mathrm{i}\theta}} \times (\mathrm{e}^{\mathrm{i}\theta}/r + 1/k)^k\,\mathrm{d}\theta}{1 - (\mathrm{e}^{\mathrm{i}\theta} \times t)/r}.$$

Since

$$|\mathrm{e}^{-r\mathrm{e}^{-\mathrm{i}\theta}}| = |\mathrm{e}^{r\cos\theta} \times \mathrm{e}^{\mathrm{i}r\sin\theta}| = \mathrm{e}^{-r\cos\theta},$$

$$\left|\frac{\mathrm{e}^{\mathrm{i}\theta}}{r} + \frac{1}{k}\right|^k = \left(\frac{1}{rk}\right)^k (k^2 + r^2 + 2rk\cos\theta)^{k/2},$$

$$\left|1 - \frac{\mathrm{e}^{\mathrm{i}\theta} \times t}{r}\right| \geqslant 1 - \frac{|t|}{r},$$

we thus find

$$\left|\int_C \frac{x^{-1}f(x^{-1})v(x)}{1-xt}\,\mathrm{d}x\right| \leqslant \left(\frac{1}{rk}\right)^k \times \frac{1}{1-|t|/r}$$
$$\times \int_0^{2\pi} \mathrm{e}^{-r\cos\theta}(k^2 + r^2 + 2rk\cos\theta)^{k/2}\,\mathrm{d}\theta.$$

Moreover

$$|\tilde{v}_k(t)| = \left|1 + \frac{t}{k}\right|^k \geqslant \tfrac{1}{2}|\mathrm{e}^t| \geqslant \tfrac{1}{2}\mathrm{e}^{-|t|}$$

(k large enough). So it follows that

$$|E_k(t)| \leqslant \frac{\mathrm{e}^{|t|}}{\pi}\left(\frac{1}{rk}\right)^k \times \frac{|t|^k}{1-|t|/r}$$
$$\times \int_0^{2\pi} \mathrm{e}^{-r\cos\theta}(k^2 + r^2 + 2rk\cos\theta)^{k/2}\,\mathrm{d}\theta$$

for $r > |t|$ and for k large enough (i.e. $k \geqslant k_0$ where k_0 is a constant which only depends on $|t|$).

By taking $r = k$ we obtain

$$|E_k(t)| \leqslant \frac{e^{|t|}}{\pi}\left(\frac{\sqrt{2}}{k}\right)^k \times \frac{|t|^k}{1-|t|/k}\int_0^{2\pi} e^{-k\cos\theta}(1+\cos\theta)^{k/2}\,d\theta.$$

Let us remark that

$$\begin{aligned} e^{-k\cos\theta}(1+\cos\theta)^{k/2} &= e^{-k\cos\theta}(2\cos^2\theta/2)^{k/2} \\ &= (\sqrt{2})^k e^{-k\cos\theta}\,|\cos\theta/2|^k \\ &= (\sqrt{2})^k (e^{-\cos\theta}|\cos\theta/2|)^k. \end{aligned}$$

Studying the function $u(\theta) = e^{-\cos\theta}|\cos\theta/2|$ on the interval $[0, 2\pi]$ the maximum of $u(\theta)$ occurs for $\theta = \frac{2}{3}\pi$. The value of u at this point is $\frac{1}{2}\sqrt{e}$ (<1). Finally we obtain

$$|E_k(t)| \leqslant 2e^{|t|}\left(\frac{\sqrt{e}}{k}\right)^k \times \frac{|t|^k}{1-|t|/k} \leqslant 4e^{|t|}\left(\frac{\sqrt{e}}{k}\right)^k \times |t|^k. \qquad \square$$

1.2.

Let us now consider the remainder term:

$$e_k(t) = e^{-t} - \sum_{j=0}^{k-1}(-t)^j/j!.$$

Let us give a short proof of the following well-known and useful (see the remark of [12, p. 588]) proposition.

Proposition 1.3. *For a fixed $t \in \mathbb{C}^*$, we have*

$$|e_k(t)| \underset{k\to\infty}{\sim} \frac{|t|^k}{k!}.$$

Proof.

$$e_k(t) = \sum_{j\geqslant k}(-t)^j/j! = \frac{(-t)^k}{k!}\sum_{j\geqslant 0}\frac{k!}{(j+k)!}(-t)^j.$$

Let

$$S_k(t) = \sum_{j\geqslant 0}\frac{k!}{(j+k)!}(-t)^j.$$

We have

$$|S_k(t) - 1| = \left|\sum_{j\geqslant k}\frac{k!}{(j+k)!}(-t)^j\right| \leqslant \sum_{j\geqslant 1}\left(\frac{|t|}{k}\right)^j.$$

If $k > 2|t|$ then $|t|/k < \frac{1}{2}$, so we get

$$|S_k(t) - 1| \leqslant \frac{|t|/k}{1 - |t|/k} \quad \text{and} \quad \lim_{k\to\infty} S_k(t) = 1.$$

The proposition is proved since $|e_k(t)| = (|t|^k/k!) \times |S_k(t)|$. □

Remarks. (a) t being fixed, the inequality $|e_k(t)| \geqslant \frac{1}{2}(|t|^k/k!)$ is true for k large enough.
(b) $\lim_{k\to\infty} |e_k(t)|^{1/k} = 0$.

1.3.

Let us now compare $|E_k(t)|$ and $|e_k(t)|$.

Theorem 1.4. *Let $t \in \mathbb{C}^*$. For every $0 < \eta < \sqrt{e}$, we have*

$$\lim_{k\to\infty} \left[\eta^k \frac{e^{-t} - (k - 1/k)(t)}{e^{-t} - \sum_{j=0}^{k-1} (-t)^j/j!} \right] = 0.$$

(This result means that the sequence $((k - 1/k)(t))_{k\geqslant 1}$ converges to e^{-t} faster than the partial sums $(\sum_{j\geqslant 0}^{k-1} (-t)^j/j!)_{k\geqslant 1}$.

Proof. By virtue of Theorem 1.2 and Proposition 1.3 the ratio $|E_k(t)|/|e_k(t)|$ is smaller than

$$4e^{|t|} \times |t|^k \times \left(\frac{\sqrt{e}}{k}\right)^k \times \frac{2k!}{|t|^k} = 8k!e^{|t|}\left(\frac{\sqrt{e}}{k}\right)^k.$$

By using Stirling's formula, $k! \sim \sqrt{(2\pi k)}\, k^k e^{-k}$ $(k \to \infty)$, we get $k!(\sqrt{e}/k)^k \underset{\infty}{\sim} \sqrt{(2\pi k)}(1/\sqrt{e})^k$ and thus $\eta^k k!(\sqrt{e}/k)^k \underset{\infty}{\sim} \sqrt{(2\pi k)}(\eta/\sqrt{e})^k$. If $0 < \eta < \sqrt{e}$, the last term converges to 0 when $k \to \infty$, and the expression $\eta^k |E_k(t)|/|e_k(t)|$ does too as desired. □

Remark. We would have been able to choose $(x + \rho^{-k})^k$ $(0 < \rho < 1)$ as the generating polynomial. By Eiermann's theorem we still have the uniform convergence of the Padé-type approximants $(k - 1/k)$ to e^{-z} on the compact subsets of $\mathbb{C}^*$. But we cannot even prove the convergence by the same majorization technique of the previous section because the rate of the term to integrate is $\exp(\rho^{-k}/k)$, which is too large.

2. Padé-type approximants of $z \to \sum_{n=0}^{+\infty} q^{n(n-1)/2} z^n$ $(0 < q < 1)$ whose denominators are $(1 - zq^k)^k$

Let us consider the power series $f(z) = \sum_{n\geqslant 0} a_n z^n$ with
(a) $a_n \neq 0$ for all $n \in \mathbb{N}$,
(b) $\lim_{n\to\infty} (a_{n+1} \times a_{n-1})/a_n^2 = q$, $q \in \mathbb{C}^*$, $|q| < 1$.
Then there exists $\rho \in \mathbb{R}$, $0 < \rho < 1$, such that $|a_{n+1}/a_n| \leqslant \rho |a_n/a_{n-1}|$. This involves $|a_{n+1}/a_n| \leqslant \rho^n |a_1/a_0|$. Thus f is holomorphic on the whole plane. Convergence results on Padé approximants for such functions were obtained in [5, 6].

2.1.

Let $\varepsilon_k(t) = f(t) - \sum_{j=0}^{k-1} a_j t^j$. Let us describe the asymptotic behaviour of the error $\varepsilon_k(t)$.

Proposition 2.1. *For a fixed $t \in \mathbb{C}^*$, we have*

$$|\varepsilon_k(t)| \underset{k\to\infty}{\sim} |a_k| \times |t|^k.$$

Proof. The proof is the same as that given for Proposition 1.3 after noticing that $|a_{j+k}/a_k| \leqslant (\rho^k \times |a_1/a_0|)^j$. $\square$

Remark. t being fixed, we get $|\varepsilon_k(t)| \geqslant \frac{1}{2}|a_k| \times |t|^k$, for k large enough.

2.2.

Let us now assume that $q = \rho$. We can remark that $\lim_{x\to\infty} f(x) = \infty$ because $f(x) > 1 + x$. Considering Padé-type approximants $(k - 1/k)$ of f with generating polynomials $(z - \rho^k)^k$ seems to be reasonable. Unfortunately we have no good majorization for $|f(z)|$. Why? The entire functions of the type $f(z) = \sum_{n\geqslant 0} q^{n(n-1)/2} z^n$ ($|q| < 1$) are of order 0. In particular they are of exponential type [4]. We could hope for a majorization like $|f(re^{i\theta})| \leqslant K e^{r\gamma(\theta)}$, where γ is a function of the argument θ. It was the case for e^{-z} with $\gamma(\theta) = -\cos\theta$. The Laplace transform of $f(z)$ is $g(s) = \sum_{n\geqslant 0} q^{n(n-1)/2} n!/s^{n+1}$. Since

$$\frac{q^{n(n+1)/2}(n+1)! \times q^{(n-1)(n-2)/2}(n-1)!}{(n! \times q^{n(n-1)/2})^2} = q\frac{n-1}{n} \underset{n\to\infty}{\longrightarrow} q,$$

the power series $\sum_{n\geqslant 0} q^{n(n-1)/2} n!\, u^n$ is still the Taylor series of an entire function of order 0. The single singularity of $g(s)$ is then 0 and we are not allowed to conclude [4, p. 312].

As we cannot use directly Lemma 1.1, we have to find a new expression of the remainder term $\xi_k(z) = f(z) - (k - 1/k)$. In fact we get more generally the following lemma.

Lemma 2.2. *Let $g(z) = \sum_{n\geqslant 0} c_n z^n$ be an entire function and R_k be the Padé-type approximant $(k - 1/k)$ of g whose generating polynomial is $v(x) = (x - 1/a)^k$ ($a \neq 0$).*

Then we have the following equality:

$$g(t) - R_k(t) = \frac{(-1)^k (t/a)^k}{(1 - t/a)^k} \sum_{n=0}^{k} c_n t^n \left(\sum_{p=0}^{n} C_k^p (-1)^p (a/t)^p \right) + \sum_{n=k+1} c_n t^n.$$

($C_k^p = \binom{k}{p}$.)

Proof. By Lemma 1.1 we get

$$\begin{aligned} g(t) - R_k(t) &= \frac{1}{2i\pi} \frac{t^k}{(1 - t/a)^k} \int_0^{2\pi} \frac{re^{-i\theta} g(re^{-i\theta})(e^{i\theta}/r - 1/a)^k}{1 - (e^{i\theta}/r) \times t} \times \frac{i}{r} e^{i\theta}\, d\theta \\ &= \frac{1}{2i\pi} \frac{t^k}{(1 - t/a)^k} \int_0^{2\pi} \sum_{n\geqslant 0} c_n r^{n-k} \frac{e^{-(n+1)\theta}(e^{i\theta} - r/a)^k}{1 - (e^{i\theta}/r) \times t}\, ie^{i\theta}\, d\theta \end{aligned}$$

and as the series converges uniformly on the interval $[0, 2\pi]$,

$$g(t) - R_k(t) = \frac{1}{2i\pi} \frac{t^k}{(1 - t/a)^k} \sum_{n \geq 0} c_n r^{n-k} \int_0^{2\pi} \frac{e^{-(n+1)\theta}(e^{i\theta} - r/a)^k}{1 - (e^{i\theta}/r) \times t} ie^{i\theta} d\theta$$

$$= \frac{1}{2i\pi} \frac{t^k}{(1 - t/a)^k} \sum_{n \geq 0} c_n r^{n-k} \int_{C_1} \frac{z^{-(n+1)}(z - r/a)^k}{1 - z \times t/r} dz,$$

where $C_1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Let

$$I_n = \int_{C_1} \frac{z^{-(n+1)}(z - r/a)^k}{1 - z \times t/r} dz,$$

we remark that

$$\frac{1}{2i\pi} I_n = \mathrm{Res}\left(z = 0, \frac{z^{-(n+1)}(z - r/a)^k}{1 - z \times t/r}\right)$$

(Res denotes the residue). Let us write the expansion

$$\frac{(z - r/a)^k}{1 - z \times t/r} = \left(\sum_{p=0}^{k} C_k^p (-1)^{k-p} \frac{r^{k-p}}{a^{k-p}} z^p\right)\left(\sum_{l \geq 0} \frac{t^l}{r^l} z^l\right)$$

$$= \sum_{n \geq 0} \left(\sum_{p=0}^{n} C_k^p (-1)^{k-p} \frac{r^{k-n}}{a^{k-p}} t^{n-p}\right) z^n.$$

So it follows

$$\frac{1}{2i\pi} I_n = \left(\sum_{p=0}^{k} C_k^p (-1)^p a^p t^{n-p}\right)(-1)^k \frac{r^{k-n}}{a^k}$$

(we set $C_k^p = 0$ for $p > k$, $C_k^p = k!/(p!(k-p)!)$ for $p \leq k$). Then

$$g(t) - R_k(t) = \frac{(-1)^k (t/a)^k}{(1 - t/a)^k} \sum_{n \geq 0} c_n t^n \left(\sum_{p=0}^{n} C_k^p (-1)^p (a/t)^p\right).$$

Moreover

$$\sum_{n \geq k+1} c_n t^n \left(\sum_{p=0}^{n} C_k^p (-1)^p (a/t)^p\right)$$

$$= \sum_{n \geq k+1} c_n t^n \left(\sum_{p=0}^{k} C_k^p (-1)^p (a/t)^p\right)$$

$$= \sum_{n \geq k+1} c_n t^n (1 - t/a)^k$$

$$= (a/t)^k (-1)^k (1 - t/a)^k \sum_{n \geq k+1} c_n t^n.$$

Finally we obtain

$$g(t) - R_k(t) = \frac{(-1)^k (t/a)^k}{(1-t/a)^k} \sum_{n=0}^{k} c_n t^n \left(\sum_{p=0}^{n} C_k^p (-1)^p (a/t)^p \right) + \sum_{n=k+1} c_n t^n. \qquad \square$$

Let us remark that

$$\sum_{n=0}^{k} c_n t^n \left(\sum_{p=0}^{n} C_k^p (-1)^p (a/t)^p \right) = \sum_{n=0}^{k} \left(\sum_{p=0}^{k-n} c_{n+p} C_k^p (-1)^p a^p \right) t^n.$$

Applying Lemma 2.2. to $f(z) = \sum_{n \geqslant 0} q^{n(n-1)/2} z^n$ implies the following theorem.

Theorem 2.3. *For $t \in \mathbb{C}^*$ fixed, we have*

$$\xi_k(t) = \frac{(-1)^k t^k}{(1-tq^k)^k} \sum_{n=0}^{k} \left(\sum_{p=0}^{k-n} (-1)^k C_k^p (q^k)^{k-p} \times q^{(p+n)(p+n-1)/2} \right) t^n + \sum_{n \geqslant k+1} q^{n(n-1)/2} t^n.$$

2.3. A new proof of convergence

Let us prove directly the second result of Theorem A starting from the expression of $\xi_k(t)$ given above.

Theorem 2.4. *For all fixed complex numbers, t, $\lim_{k \to \infty} |\xi_k(t)|^{1/k} = 0$.*

Proof. We have

$$\left| \sum_{p=0}^{k-n} (-1)^p C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2} \right| \leqslant \sum_{p=0}^{k-n} C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2}$$

and

$$\sum_{p=0}^{k-n} C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2} = q^{k(k-1)/2} \sum_{p=0}^{k-n} C_k^p q^{\psi_n(p)}$$

$(0 \leqslant n \leqslant k)$ where we set

$$p \in [0, k-n], \quad \psi_n(p) = \tfrac{1}{2}k^2 + \tfrac{1}{2}k - pk + \tfrac{1}{2}(p+n)(p+n-1).$$

Since $\psi_n'(p) = -k + n - \frac{1}{2} + p$, ψ_n is decreasing throughout the whole interval $[0, k-n]$. We also have $\psi_n(0) = \frac{1}{2}k^2 + \frac{1}{2}k + \frac{1}{2}n(n-1)$ and $\psi_n(k-n) = nk$. It follows $\forall p \in [0, k-n]$, $nk \leqslant \psi_n(p) \leqslant \frac{1}{2}k^2 + \frac{1}{2}k + \frac{1}{2}n(n-1)$.

Thus

$$\sum_{p=0}^{k-n} C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2} \leqslant q^{k(k-1)/2} \sum_{p=0}^{k-n} C_k^p (q^k)^n$$

$$\leqslant 2^k q^{k(k-1)/2} (q^k)^n.$$

Then

$$\left|\sum_{n=0}^{k}\left(\sum_{p=0}^{k-n}(-1)^p C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2}\right)t^n\right|$$
$$\leqslant 2^k q^{k(k-1)/2}\sum_{n=0}^{k}(q^k|t|)^n$$
$$\leqslant 2^k q^{k(k-1)/2}/(1-q^k|t|).$$

Finally we obtain

$$|\xi_k(t)|^{1/k} \leqslant 4q^{(k-1)/2}/(1-q^k|t|)^{1/k} + 2^{1/k}|t|^{1+1/k}q^{(k+1)/2}$$

and thus $\lim_{k\to\infty}|\xi_k(t)|^{1/k} = 0$. □

2.4.

Let us now deal with the ratio $|\xi_k(t)|/|\varepsilon_k(t)|$. According to Proposition 2.1 and Theorem 2.3 its behaviour is asymptotically equivalent to

$$\left|\sum_{n=0}^{k}\left(\sum_{p=0}^{k-n}(-1)^k C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2}\right)t^n\right| \Big/ q^{k(k-1)/2}$$

$(k\to\infty)$. We have

$$\sum_{n=0}^{k}\left(\sum_{p=0}^{k-n}(-1)^k C_k^p (q^k)^{k-p} q^{(p+n)(p+n-1)/2}\right)t^n$$
$$= q^{k(k-1)/2}\sum_{n=0}^{k}\left(\sum_{p=0}^{k-n}(-1)^p C_k^p q^{\psi_n(p)}\right)t^n.$$

If $n \geqslant 1$, $q^{\psi_n^{(p)-k}} \leqslant (q^k)^{n-1} \leqslant 1$, so it follows

$$\left|\sum_{n=1}^{k}\left(\sum_{p=0}^{k-n}(-1)^p C_k^p q^{\psi_n(p)}\right)t^n\right| \leqslant q^k \sum_{n=1}^{k}\left(\sum_{p=0}^{n-k} C_k^p (q^k)^{n-1}\right)|t|^n$$
$$\leqslant 2^k q^k|t|/(1-q^k|t|).$$

If $0<q<\frac{1}{2}$ the last term tends to 0. Thus, if $0<q<\frac{1}{2}$ we have

$$|\xi_k(t)|/|\varepsilon_k(t) = |\mathrm{o}(1)+\mu_k|,$$

where

$$\mu_k \underset{\infty}{\sim} \sum_{p=0}^{k} C_k^p(-1)^p q^{\psi_0(p)} \quad \left(=(-1)^k\sum_{p=0}^{k} C_k^p(-1)^p q^{p(p+1)/2}\right).$$

If we use the usual notations of successive differences, we get $\mu_k \underset{\infty}{\sim} \Delta^k y_0$ with $y_p = q^{p(p+1)/2}$. Unfortunately we were not able to give the behaviour of μ_k. But by numerical experiments $|\mu_k|$ seems to tend to infinity. Finally let us conjecture that, if $0<q<\frac{1}{2}$, the partial sums of the Taylor series converge faster to $\sum_{n=0}^{k+\infty} q^{n(n-1)/2} z^n$ than the sequence of Padé-type approximants.

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References

[1] C. Brezinski, *Padé Type Approximation and General Orthogonal Polynomials*, Internat. Ser. Numer. Math. **50** (Birkhäuser, Basel, 1980).
[2] C. Brezinski and J. Van Iseghem, Padé approximations, in: P.G. Ciarlet and J.L. Lions, Eds., *Handbook of Numerical Analysis, Vol. 3* (North-Holland, Amsterdam, 1992) 47–222.
[3] M. Eiermann, On the convergence of Padé-type approximants to analytic functions, *J. Comput. Appl. Math.* **10** (1984) 219–227.
[4] P. Henrici, *Applied and Computational Complexe Analysis, Vol. 2* (Wiley, New York, 1977).
[5] D.S. Lubinsky, Uniform convergence of rows of the Padé table, *Constr. Approx.* **3** (1987) 307–330.
[6] D.S. Lubinsky and E.B. Saff, Padé tables of entire functions of very slow and smooth growth II, *Constr. Approx.* **4** (1988) 321–339.
[7] S.P. Nørsett, Restricted Padé approximations to the exponential function, *SIAM J. Numer. Anal.* **15** (1978) 1008–1029.
[8] S.P. Nørsett and A. Wolfbrandt, Attainable order of rational approximations to exponential function with only real poles, *BIT* **17** (1977) 200–208.
[9] E.B. Saff, A. Schöhage and R.S. Varga, Geometric convergence to $\exp(-z)$ by rational functions with real poles, *Numer. Math.* **25** (1978) 307–322.
[10] J.L. Siemienuch, Properties of certain rational approximants to $\exp(-z)$, *BIT* **16** (1976) 172–191.
[11] J. Van Iseghem, Padé-type approximants of $\exp(-z)$ whose denominators are $(1 + z/n)^n$, *Numer. Math.* **43** (1984) 282–292.
[12] P. Wild, Accelerating the convergence of power series of certain entire functions, *Numer. Math.* **51** (1987) 583–595.